Effect of signature change in NGR

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The field equations of the new general relativity (NGR) have been applied to an absolute parallelism space having three unknown functions of the radial coordinate. The field equations have been solved using two different schemes. In the first scheme, we used the conventional procedure used in orthodox general relativity. In the second scheme, we examined the effect of signature change. The latter scheme gives a solution which is different from the Schwarzschild one. In both methods we find solution of the field equations under the same constraint imposed on the parameters of the theory. We also calculated the energy associated with the solutions in the two cases using the superpotential method. We found that the energetic content of one of the solutions is different from that of the other. A comparison between the two solutions obtained in the present work and a third one obtained by Hayashi and Shirafuji (1979) shows that the change of the signature may give rise to new physics.

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1. Introduction

It is generally accepted that a successful theory of gravity should be a metric one. In applications of such a theory, it is necessary to attribute a property called signature to the metric representing space-time (cf. [1]). The signature usually chosen, in most applications, is the Lorentz signature (corresponding to an indefinite metric). This signature guarantees a desirable result that any gravity theory of this class will have correct special relativistic limits. However, it appears that there is a need for another signature different from the Lorentz one, giving rise to a positive definite metric, especially when dealing with some problems connected to quantum cosmology [2]. It is well known that a complete consistent theory that combines quantum mechanics and gravity is still beyond the reach of researchers. However, one important features of such a theory, if it exists, is that it should incorporate Feynman's proposal to formulate quantum theory in terms of a sum over histories. Severe technical problems arise when one tries to apply Feynman's proposal in this context. The only way to overcome these problems is that one must add up waves for particle histories not in the real time but in the imaginary time [3]. This necessitates the use of a positive-definite metric and not an indefinite one. So, it is of interest to examine the effect of the signature of the metric on solution of different geometric theories of gravity.

Some authors started to examine the effect of signature change on the solutions of the field equations of general relativity [4, 5, 6, 7, 8, 9]. The attempts of those authors can be classified in two different classes:
The first class, The main assumption in this class is that the signature of the metric is changed, from a positive-definite metric to an indefinite one, at a very early time in the history of the universe. This means that the change is represented as a jump process across a certain hypersurface (cf.[5]).
The second class, This class depends on a different assumption. That is, the space-time has 4-dimensions with a positive-definite metric, but measurements are carried out in (3+1)-dimensions with an indefinite metric giving rise to Lorentz signature [4]. Consequently, Lorentz signature is to be imposed on the metric just before matching the results of the theory with measurements. In other words, the field equations of the theory are to be solved using a positive-definite metric and then, after obtaining the solution, Lorentz signature is to be imposed on the metric.

The effects of signature change were examined, in the previous mentioned attempts, in the case of general relativity. It is of interest to examine these effects on the solutions of other metric field theories different from general relativity. Geometric field theories of this class are those theories written in absolute parallelism spaces (cf.[10, 11, 12]). Recently, more attention is paid to such theories since they include non-vanishing torsion which is connected to Dirac field [13] and to string theory [14]. One of those theories is the New General Relativity (NGR) constructed by Hayashi and Shirafuji [12]. The basic geometry for this theory is the absolute parallelism (AP) geometry, in which a metric can always be defined.

The aim of the present work is to study the effects of signature change on the spherically symmetric solutions of the NGR and to calculate the energy associated with each solution, using the superpotential method of Mikhail et al. [15] is calculated. In section

2 we briefly review the field equations of NGR and the most general AP-structure having spherical symmetry, used for applications. To explore the effect of signature change on the solution, we are going, in section 3, to solve the field equations for the spherically symmetric structure, in two different cases. *Case I:* by imposing Lorentz signature , on the metric, from the beginning, i.e., before solving the field equations; *Case II:* we solve the field equation; using a positive-definite metric then we impose Lorentz signature to the solution obtained. In section 4 the energy associated with each solution is calculated using the superpotential method. We compare and discuss the results of section 3 and 4 in section 5.

2. NGR Field Equations and Geometric Structure

In 1979 Hayashi and Shirafuji constructed a theory which they called "The New General Relativity", (NGR). They have used an AP- space for its formulation, with the field variables being the 16 tetrad components $(\underset{i}{\lambda}^{\mu})^*$. Assuming invariance under:

a) the group of general coordinate transformations, and

b) the group of global Lorentz transformations.

They wrote the general gravitational Lagrangian density quadratic in the torsion tensor as[†]

$$\mathcal{L}_G = \sqrt{-g}\left[\frac{R}{2\kappa} + d_1(t^{\mu\nu\lambda}t_{\mu\nu\lambda}) + d_2(C^{\mu}C_{\mu}) + d_3(a^{\mu}a_{\mu}) - d_4(C^{\mu}a_{\mu})\right], \tag{1}$$

where d_1, d_2, d_3 and d_4 are dimensionless parameters of the theory [‡], and

$$\begin{aligned}
t_{\mu\nu\lambda} &\stackrel{\text{def.}}{=} \Lambda_{(\mu\nu)\lambda} - \frac{1}{3}g_{\lambda(\mu}C_{\nu)} + \frac{1}{3}g_{\mu\nu}C_{\lambda}, \\
\Lambda^{\lambda}{}_{\mu\nu} &\stackrel{\text{def.}}{=} \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = \underset{i}{\lambda}^{\lambda}(\underset{i}{\lambda}_{\mu,\nu} - \underset{i}{\lambda}_{\nu,\mu}), \qquad (\text{Torsion tensor}), \\
\Gamma^{\lambda}{}_{\mu\nu} &\stackrel{\text{def.}}{=} \underset{i}{\lambda}^{\lambda}\underset{i}{\lambda}_{\mu,\nu}, \qquad (\text{Nonsymmetric connection}), \\
C_{\mu} &\stackrel{\text{def.}}{=} \Lambda^{\lambda}{}_{\mu\lambda}, \qquad (\text{Basic vector}), \\
a_{\mu} &\stackrel{\text{def.}}{=} \frac{1}{6}\epsilon_{\mu\nu\lambda\rho}\Lambda^{\nu\lambda\rho}, \qquad (\text{Axial vector}), \\
\epsilon_{\mu\nu\lambda\rho} &\stackrel{\text{def.}}{=} \sqrt{-g}\delta_{\mu\nu\lambda\rho}, \qquad (\text{Completely antisymmetric}), \\
g_{\mu\nu} &\stackrel{\text{def.}}{=} \underset{i}{\lambda}_{\mu}\underset{i}{\lambda}_{\nu}, \qquad (\text{Metric tensor}),
\end{aligned} \tag{2}$$

$\delta_{\mu\nu\lambda\rho}$ is the Levi-Civita antisymmetric tensor and R is the Ricci scalar.

*Latin indices $(i, j, k, \cdots)$ designate the vector number, which runs from 0 to 3, while Greek indices $(\mu, \nu, \rho, \cdots)$ designate the world-vector components running from 0 to 3. Latin indices $(a, b, c, \cdots)$, and Greek indices $(\alpha, \beta, \gamma, \cdots)$, run from 1 to 3.

† Throughout this paper we use the relativistic units, $c = G = 1$. The Einstein constant κ is then 8π. We will denote the symmetric part by the parentheses ( ), for example, $A_{(\mu\nu)} \stackrel{\text{def.}}{=} (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the brackets [ ], $A_{[\mu\nu]} \stackrel{\text{def.}}{=} (1/2)(A_{\mu\nu} - A_{\nu\mu})$.

‡ The dimensionless parameters c_i of Ref.[12] are here denoted by d_i for convenience.

By applying the variational principle to the Lagrangian (1), they obtained the field equation:

$$I^{\mu\nu} = \kappa T^{\mu\nu}, \tag{3}$$

where,

$$I^{\mu\nu} \stackrel{\text{def.}}{=} G^{\mu\nu} + 2\kappa\Big[D_\lambda F^{\mu\nu\lambda} - C_\lambda F^{\mu\nu\lambda} + H^{\mu\nu} - \frac{1}{2}g^{\mu\nu}L'\Big], \tag{4}$$

and

$$F^{\mu\nu\lambda} \stackrel{\text{def.}}{=} \frac{1}{2}\underset{i}{\lambda}^{\mu}\frac{\partial L_G}{\partial \underset{i}{\lambda}_{\nu,\lambda}} = -F^{\mu\lambda\nu}, \tag{5}$$

$$H^{\mu\nu} \stackrel{\text{def.}}{=} \Lambda^{\rho\sigma\mu}F_{\rho\sigma}{}^{\nu} - \frac{1}{2}\Lambda^{\nu\rho\sigma}F^{\mu}{}_{\rho\sigma} = H^{\nu\mu}, \tag{6}$$

$$L' \stackrel{\text{def.}}{=} \Big[d_1(t^{\mu\nu\lambda}t_{\mu\nu\lambda}) + d_2(C^\mu C_\mu) + d_3(a^\mu a_\mu) - d_4(C^\mu a_\mu)\Big], \tag{7}$$

$$T^{\mu\nu} \stackrel{\text{def.}}{=} \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta\underset{i}{\lambda}_{\nu}}\underset{i}{\lambda}^{\mu}. \tag{8}$$

Here $\mathcal{L}_M$ denotes the Lagrangian density of material fields, of which the energy-momentum tensor $T^{\mu\nu}$ is nonsymmetric in general.

In the case of *static*, *spherically symmetric* space-time, with tetrad vector fields having a diagonal form, the field equations (4) have been exactly solved [12, 16]. The exact solution obtained is the same as that obtained by assuming the invariance under parity operation [22].

In order to reproduce the correct Newtonian limit, the parameters d_1 and d_2 should satisfy the condition [12]

$$d_1 + 4d_2 + 9d_1d_2 = 0, \tag{9}$$

which is called the Newtonian approximation condition. This condition can be satisfied by taking,

$$d_1 = -\frac{1}{3(1-\epsilon)}, \quad d_2 = \frac{1}{3(1-4\epsilon)}, \tag{10}$$

where ϵ is a dimensionless parameter. Comparison with Solar - System observations indicates that $|\epsilon|$ must be very small.

The structure of the AP-spaces with spherical symmetry has been studied by Robertson [18]. The tetrad vectors defining completely this structure in Cartesian coordinates can be written as

$$\begin{aligned}
\underset{0}{\lambda}^{0} &= A(r),\\
\underset{0}{\lambda}^{\alpha} &= D(r)X^\alpha,\\
\underset{a}{\lambda}^{0} &= E(r)X^a\\
\underset{a}{\lambda}^{\alpha} &= \delta_a{}^\alpha B(r) + F(r)X^aX^\alpha + \epsilon_{a\alpha c}S(r)X^c,
\end{aligned} \tag{11}$$

where $A(r)$, $D(r)$, $E(r)$, $F(r)$, $B(r)$ and $S(r)$ are functions of the radial coordinate $r = (X^aX^a)^{1/2}$. It has been shown that [18] :

1) Improper rotation is admitted if $S(r)$=0.

2) The functions $E(r)$ and $F(r)$ can be eliminated by mere coordinate transformations, leaving the tetrad in the simple form

$$\left.\begin{aligned} \underset{0}{\lambda}{}^{0} &= A(r), \\ \underset{0}{\lambda}{}^{\alpha} &= D(r)X^{\alpha}, \\ \underset{a}{\lambda}{}^{\alpha} &= \delta_{a}{}^{\alpha}B(r). \end{aligned}\right\} \tag{12}$$

It is of interest to point out that the tetrad vectors used previously to obtain an exact solution [12] is a special case of the tetrad (12) when the function $D(r) = 0$. Thus one may expect to obtain more general solutions when the tetrad (12) is applied to the field equations (4). The tetrad (12) can be written in the form:

$$\left(\underset{i}{\lambda}{}^{\mu}\right) = \begin{pmatrix} A(r) & D(r)r & 0 & 0 \\ 0 & B(r)\sin\theta\cos\phi & \dfrac{B(r)}{r}\cos\theta\cos\phi & -\dfrac{B(r)\sin\phi}{r\sin\theta} \\ 0 & B(r)\sin\theta\sin\phi & \dfrac{B(r)}{r}\cos\theta\sin\phi & \dfrac{B(r)\cos\phi}{r\sin\theta} \\ 0 & B(r)\cos\theta & -\dfrac{B(r)}{r}\sin\theta & 0 \end{pmatrix}, \tag{13}$$

in spherical polar coordinates (t, r, θ, ϕ). Consequently, the metric tensor of the Riemannian space, associated with the AP-space (13) can be written in the form (using (2))

$$\begin{aligned} &g_{00} = \frac{(B(r)^2 + D(r)^2r^2)}{A(r)^2B(r)^2}, \quad g_{01} = g_{10} = -\frac{D(r)r}{A(r)B(r)^2}, \quad g_{11} = -\frac{1}{B(r)^2}, \\ &g_{22} = -\frac{r^2}{B(r)^2}, \quad g_{33} = \frac{r^2\sin\theta^2}{B(r)^2}. \end{aligned} \tag{14}$$

3. Solution Of The Field Equation

For later convenience, we will redefine the functions A and D such that $A \rightarrow c^{\star}A$ and $D \rightarrow c_1{}^{\star}D$. This will not affect the geometric structure (13). Now substituting into the field equations (4) using (13), we get the following set of differential equations ($c^{\star}$ and $c_1{}^{\star}$ are parameters)

$$(-1+\kappa(d_1+4d_2))c^{\star 2}A\Bigg\{B^2\Bigg(2\epsilon\left(\frac{A'}{A}\right)'+2(1-2\epsilon)\left(\frac{B'}{B}\right)'+\frac{4}{r}\left[\epsilon\frac{A'}{A}+(1-2\epsilon)\frac{B'}{B}\right]-2\epsilon\frac{A'B'}{AB}$$
$$-\epsilon\left(\frac{A'}{A}\right)^2-(1-4\epsilon)\left(\frac{B'}{B}\right)^2-3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\Bigg)+c_1^{\star 2}D^2r\Bigg[l(r)-r\Bigg(s(r)+b(r)\Bigg)\Bigg]\Bigg\}=0, \quad (15)$$

$$(-1+\kappa(d_1+4d_2))c^{\star}c_1^{\star}AD\Bigg\{B^2\Bigg(2(1-2\epsilon)\frac{A'}{A}+2\frac{B'}{B}-r\Bigg(2(1-2\epsilon)\frac{A'B'}{AB}+\epsilon\left(\frac{A'}{A}\right)^2+\left(\frac{B'}{B}\right)^2$$
$$+3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\Bigg)\Bigg)-c_1^{\star 2}D^2r^2\Bigg[r\Bigg(s(r)+b(r)\Bigg)-l(r)\Bigg]\Bigg\}=0, \quad (16)$$

$$(-1+\kappa(d_1+4d_2))c^{\star}c_1^{\star}AD\Bigg\{B^2\left[2(1-2\epsilon)\frac{A'}{A}+2(1-3\epsilon)\frac{B'}{B}+8\epsilon\frac{D'}{D}\right]-B^2r\Bigg(2(1-3\epsilon)\frac{A'B'}{AB}$$
$$+\epsilon\left(\frac{A'}{A}\right)^2+(1-6\epsilon)\left(\frac{B'}{B}\right)^2+2\epsilon\frac{A'D'}{AD}+2\epsilon\frac{B''}{B}+6\epsilon\frac{B'D'}{BD}-2\epsilon\frac{D''}{D}+3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\Bigg)$$
$$-c_1^{\star 2}D^2r^2\Bigg[r\Bigg(s(r)+b(r)\Bigg)-l(r)\Bigg]\Bigg\}=0, \quad (17)$$

$$(-1+\kappa(d_1+4d_2))\Bigg\{B^2c_1^{\star 2}D^2r\left[2(2-3\epsilon)\frac{A'}{A}+2(4-3\epsilon)\frac{B'}{B}-2(1-\epsilon)\frac{D'}{D}\right]$$
$$-B^2c_1^{\star 2}D^2r^2\Bigg(2(2-3\epsilon)\frac{A'B'}{AB}+2\epsilon\left(\frac{A'}{A}\right)^2+(4-3\epsilon)\left(\frac{B'}{B}\right)^2-2\epsilon\frac{A'D'}{AD}-2(1-\epsilon)\frac{B'D'}{BD}+\epsilon\left(\frac{D'}{D}\right)^2$$
$$+3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\Bigg)+B^4\Bigg[2(1-2\epsilon)\frac{A'B'}{AB}+\epsilon\left(\frac{A'}{A}\right)^2+\left(\frac{B'}{B}\right)^2-\frac{2}{r}\left((1-2\epsilon)\frac{A'}{A}+\frac{B'}{B}\right)$$
$$+3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\Bigg]+c_1^{\star 4}D^4r^3\Bigg[r\Bigg(s(r)+b(r)\Bigg)-l(r)\Bigg]\Bigg\}=0, \quad (18)$$

$$(-1+\kappa(d_1+4d_2))B^2\Bigg\{c_1^{\star 2}D^2\Bigg[(1-2\epsilon)\frac{A''}{A}-3(1-2\epsilon)\frac{A'B'}{AB}-(2-5\epsilon)\left(\frac{A'}{A}\right)^2-5(1-\epsilon)\left(\frac{B'}{B}\right)^2$$
$$+(3-8\epsilon)\frac{A'D'}{AD}-2(1-2\epsilon)\frac{B''}{B}+(5-8\epsilon)\frac{B'D'}{BD}-(1-2\epsilon)\frac{D''}{D}-(1-3\epsilon)\left(\frac{D'}{D}\right)^2\Bigg]$$

$$+\frac{c_1^{\star 2}D^2}{r}\left(4(1-2\epsilon)\frac{A'}{A}+8(1-\epsilon)\frac{B'}{B}-2(3-5\epsilon)\frac{D'}{D}\right)+\frac{B^2}{r^2}\left((1-2\epsilon)\frac{A''}{A}-2\epsilon\frac{A'B'}{AB}-(2-5\epsilon)\left(\frac{A'}{A}\right)^2\right.$$
$$\left.\left.-\left(\frac{B'}{B}\right)^2+\frac{B''}{B}+\frac{1}{r}\left[(1-2\epsilon)\frac{A'}{A}+\frac{B'}{B}\right]-3\frac{(1-\epsilon)c_1^{\star 2}D^2}{B^2}\right)\right\}=0, \tag{19}$$

where,

$$\begin{aligned}
\epsilon &\overset{\text{def.}}{=} \frac{\kappa(d_1+d_2)}{(-1+\kappa(d_1+4d_2))},\\
l(r) &\overset{\text{def.}}{=} 2(1-\epsilon)\left[\frac{A'}{A}+3\frac{B'}{B}-\frac{D'}{D}\right],\\
s(r) &\overset{\text{def.}}{=} \left[\epsilon\left(\frac{A'}{A}\right)^2+3(1-\epsilon)\left(\frac{B'}{B}\right)^2+\epsilon\left(\frac{D'}{D}\right)^2\right],\\
b(r) &\overset{\text{def.}}{=} 2\left[(1-\epsilon)\frac{A'B'}{AB}-\epsilon\frac{A'D'}{AD}-(1-\epsilon)\frac{B'D'}{BD}\right],
\end{aligned} \tag{20}$$

and $A' \overset{\text{def.}}{=} \frac{dA}{dr}$, $\quad B' \overset{\text{def.}}{=} \frac{dB}{dr}$ and $\quad D' \overset{\text{def.}}{=} \frac{dD}{dr}$.

We are going to find a general solution of the differential equations (15)-(19) following the method given by Mazunder and Ray [19] for two cases, i.e, the indefinite and positive-definite cases.

Case I: indefinite metric

In this case we take the value of $c^\star = c_1^\star = \sqrt{-1}$. Using (15), (16) (taking $\epsilon = 0$, to simplify the equations) we get

$$(rB'-B)BA'+(BB''-B'^2)rA+ABB'=0, \tag{21}$$

equation (21) can be integrated to give the function A, in terms of the unknown function B, in the form

$$A=\frac{k_1}{\left(1-\frac{rB'}{B}\right)}, \tag{22}$$

k_1 being a constant of integration.

From (15), using (22), we get after some rearrangements:

$$\begin{aligned}
&D^2\left(2BB''r^3-5B'^2r^3-3rB^2+8r^2BB'\right)+DD'\left(2BB'r^3-2r^2B^2\right)\\
&+2rB^3B''-3rB'^2B^2+4B^3B'=0.
\end{aligned} \tag{23}$$

Using the transformation,

$$B=e^\alpha, \quad z=ln\ r, \quad D^2=\beta, \tag{24}$$

then (23) will give,

$$\left(2\alpha_{zz}-3{\alpha_z}^2+6\alpha_z-3\right)+\frac{\beta_z}{\beta}\left(\alpha_z-1\right)-\frac{1}{\beta}e^{(2\alpha-2z)}\left(2\alpha_{zz}-{\alpha_z}^2+2\alpha_z\right)=0, \tag{25}$$

where $\alpha_z = \frac{d\alpha}{dz}$. Now the solution of (25) can be written in the form

$$\beta = D^2 = \frac{e^{3(\alpha - z)}}{(1-\alpha_z)^2}\left\{k_2 + \alpha_z(\alpha_z - 2)e^{(z-\alpha)}\right\}, \tag{26}$$

k_2 being another constant of integration. Using (24), we rewrite (26) in the form

$$D^2 = \frac{1}{\left(1 - \frac{rB'}{B}\right)^2}\left(\frac{B}{r}\right)^3\left\{k_2 + \frac{rB'}{B}\left(\frac{rB'}{B} - 2\right)\frac{r}{B}\right\}, \tag{27}$$

in terms of the arbitrary function B. Hence we obtain the general solution of the field equations (15)-(19) in the case $c^\star = {c_1}^\star = \sqrt{-1} = i$ in terms of an arbitrary function B for the special case $\epsilon = 0$.

The line - element in this case is given by

$$dS^2 = -\frac{(B^2 - D^2 r^2)}{A^2 B^2}dt^2 - 2\frac{Dr}{AB^2}drdt + \frac{1}{B^2}\left(dr^2 + r^2 d\Omega^2\right), \tag{28}$$

where

$$d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2. \tag{29}$$

Using the coordinate transformation [20]

$$dT = dt + \frac{DrA}{B^2 - D^2 r^2}dr, \quad \xi = \frac{(B^2 - D^2 r^2)}{A^2 B^2}, \quad \textit{and} \quad R = \frac{r}{B}, \tag{30}$$

then we can eliminate the cross term in (28), and we finally get

$$dS^2 = -\xi dT^2 + \frac{1}{\xi}dR^2 + R^2 d\Omega^2. \tag{31}$$

Using (22), (27), the terms containing the derivatives of the arbitrary function B cancel out and we finally get,

$$\xi(R) = \left(1 - \frac{k_2}{R}\right). \tag{32}$$

Taking $k_2 = 2m$, then (31) will give rise to the Schwarzschild metric. Thus in this case, i.e. indefinite metric, we get nothing more than the Schwarzschild field as a solution of the NGR field equations.

The tetrad (13) in *Case I* has been subject to two steps of coordinate transformations from (t, r, θ, ϕ) to (T, R, θ, ϕ). We now apply a further transformation from (T, R, θ, ϕ) to

the Cartesian coordinate (T, X^a) with $a = 1, 2$, and 3. The tetrad can be shown to have the form

$$
\begin{aligned}
\lambda_{0}{}^{0} &= \frac{(1 - rB')}{\left(1 - \dfrac{2m}{r}\right)}, \\
\lambda_{0}{}^{\alpha} &= \frac{x^\alpha}{r^{3/2}}\sqrt{2m + B'^2 r^3 - 2B'r^2}, \\
\lambda_{a}{}^{0} &= \frac{ix^\alpha}{r}\frac{\sqrt{2m + B'^2 r^3 - 2r^2 B'}}{(r - 2m)}, \\
\lambda_{a}{}^{\alpha} &= -i\frac{(B' x^\alpha x^a - r)}{r}.
\end{aligned} \tag{33}
$$

Tetrad (33) will be used latter in the calculations of the energy for the case of indefinite metric.
Case II: positive definite

As we mentioned before, in this case we are going to postpone the insertion of Lorentz signature till we find the solution of the field equations (15)-(19). This can be achieved by taking $c^\star = c_1^\star = 1$. Following the same procedure as in *Case I*, (taking $\epsilon = 0$), we get from (15) and (16)

$$
(rB' - B)BA' + (BB'' - B'^2)rA + ABB' = 0, \tag{34}
$$

which can be integrated to give

$$
A = \frac{\tilde{K}_1}{\left(1 - \dfrac{rB'}{B}\right)}, \tag{35}
$$

$\tilde{K}_1$ being a constant of integration. Using (15) and (35) we get after some rearrangement,

$$
\begin{aligned}
&D^2\left\{2BB''r^3 - 5B'^2r^3 - 3rB^2 + 8r^2BB'\right\} \\
&+DD'\left(2BB'r^3 - 2r^2B^2\right) + 2rB^3B'' - 3rB'^2B^2 + 4B^3B' = 0.
\end{aligned} \tag{36}
$$

Using transformation (24), we get

$$
(2\alpha_{zz} - 3{\alpha_z}^2 + 6\alpha_z - 3) + \frac{\beta_z}{\beta}(\alpha_z - 1) + \frac{1}{\beta}e^{(2\alpha - 2z)}(2\alpha_{zz} - {\alpha_z}^2 + 2\alpha_z) = 0, \tag{37}
$$

where α_z is defined above. The solution of (37) is given by,

$$
\beta = D^2 = \frac{e^{3(\alpha - z)}}{(1 - \alpha_z)^2}\left\{\tilde{K}_2 - \alpha_z(\alpha_z - 2)e^{(z - \alpha)}\right\}, \tag{38}
$$

where $\tilde{K}_2$ is another constant of integration. Hence the general solution, having spherical symmetry, of the system (15)-(19) is given by,

$$
A = \frac{\tilde{K}_1}{\left(1 - \dfrac{rB'}{B}\right)},
$$

$$D^2 = \frac{1}{\left(1-\frac{rB'}{B}\right)^2}\left(\frac{B}{r}\right)^3\left\{\tilde{K_2}-\frac{rB'}{B}\left(\frac{rB'}{B}-2\right)\frac{r}{B}\right\}. \tag{39}$$

It is clear from (39) that as $r \to \infty$, $\tilde{K_1}=1$. Since B is an arbitrary function, we are going to consider the following case.

If we take the arbitrary function B in the form

$$B = \frac{1}{\left(1+\frac{c_1}{r^2}\right)^{1/2}}, \tag{40}$$

where c_1 is a constant, then substituting (40) in (39), we get

$$\begin{aligned} A &= \left(1+\frac{c_1}{r^2}\right) \\ D^2 &= \frac{\tilde{K_2}(r^2+c_1)^{3/2}+c_1(2r^2+c_1)}{r^4(r^2+c_1)}. \end{aligned} \tag{41}$$

It is of interest to note that a solution similar to that given by (40) and (41) was obtained before by one of the authors [20] in case of the generalized field theory. From now on, we will take into account Lorentz signature for the solution (40) and (41). Using (14) the line-element, in which Lorentz signature is inserted can be written in the form,

$$dS^2 = -\frac{(B^2-D^2r^2)}{A^2B^2}dt^2 - 2\frac{Dr}{AB^2}drdt + \frac{1}{B^2}\left(dr^2+r^2d\Omega^2\right). \tag{42}$$

We can eliminate the cross term from (42), by using the coordinate transformation (30),

$$d\tilde{T} = dt - \frac{DrA}{(B^2-D^2r^2)}dr, \quad \eta = \frac{(B^2-D^2r^2)}{A^2B^2}, \quad and \quad \tilde{R} = \frac{r}{B}, \tag{43}$$

then we finally get

$$dS^2 = -\eta d\tilde{T}^2 + \frac{1}{\eta}d\tilde{R}^2 + \tilde{R}^2 d\Omega^2. \tag{44}$$

Using (40) and (41), we get

$$\eta(\tilde{R}) = \left(1-\frac{\tilde{K_2}}{\tilde{R}}-\frac{4c_1}{\tilde{R}^2}+\frac{2{c_1}^2}{\tilde{R}^4}\right), \tag{45}$$

which is different from Schwarzschild metric.

Tetrad (13) in *Case II* has been subject to two steps of coordinate transformations from (t,r,θ,ϕ) to $(\tilde{T},\tilde{R},\theta,\phi)$. We now apply a further transformation from $(\tilde{T},\tilde{R},\theta,\phi)$ to the

Cartesian coordinate $(\tilde{T}, X^a)$ with $a = 1, 2$, and 3. The tetrad can be shown to have the form

$$
\begin{aligned}
\underset{0}{\lambda}{}^{0} &= \frac{(1-\dfrac{c_1}{r^2})}{(1-\dfrac{k_2}{r}-\dfrac{4c_1}{r^2}+\dfrac{2c_1^2}{r^4})}, \\
\underset{0}{\lambda}{}^{\alpha} &= -\frac{x^\alpha}{r^{3/2}}\left(k_2+\frac{2c_1}{r}-\frac{c_1^2}{r^3}\right), \\
\underset{a}{\lambda}{}^{0} &= -i\frac{x^\alpha}{r^{3/2}}\frac{\left(k_2+\dfrac{2c_1}{r}-\dfrac{c_1^2}{r^3}\right)}{\left(1-\dfrac{k_2}{r}-\dfrac{4c_1}{r^2}+\dfrac{2c_1}{r^4}\right)}, \\
\underset{a}{\lambda}{}^{\alpha} &= -i\frac{(c_1 x^\alpha x^a - r^4)}{r^4}.
\end{aligned} \tag{46}
$$

Tetrad (46) in its Cartesian form will be used in the next section in the calculation of the energy for the positive-definite case. It is of interest to note that the Schwarzschild metric can be obtained in this case upon taking $c_1 = 0$.

4. Calculations of Energy

In this section we are going to calculate the energy associated with the two solutions obtained in the previous section using the superpotential method given by Mikhail [15].

The superpotential of Møller theory [11] is given by Mikhail et al. [15] as

$$
\mathcal{U}_\mu{}^{\nu\lambda} \stackrel{\text{def.}}{=} \frac{\sqrt{-g}}{2\kappa} P_{\chi\rho\sigma}{}^{\tau\nu\lambda}\left[C^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\mu\tau}\gamma^{\chi\rho\sigma} - (1-2\lambda) g_{\mu\tau}\gamma^{\sigma\rho\tau}\right], \tag{47}
$$

where $P_{\chi\rho\sigma}{}^{\tau\nu\lambda}$ is

$$
P_{\chi\rho\sigma}{}^{\tau\nu\lambda} \stackrel{\text{def.}}{=} \delta_\chi{}^\tau g_{\rho\sigma}{}^{\nu\lambda} + \delta_\rho{}^\tau g_{\sigma\chi}{}^{\nu\lambda} - \delta_\sigma{}^\tau g_{\chi\rho}{}^{\nu\lambda}, \tag{48}
$$

with $g_{\rho\sigma}{}^{\nu\lambda}$ being a tensor defined by

$$
g_{\rho\sigma}{}^{\nu\lambda} \stackrel{\text{def.}}{=} \delta_\rho{}^\nu \delta_\sigma{}^\lambda - \delta_\sigma{}^\nu \delta_\rho{}^\lambda, \tag{49}
$$

and $\gamma^{\chi\rho\sigma}$ is the contorsion defined by

$$
\gamma_{\mu\nu\rho} \stackrel{\text{def.}}{=} \underset{i}{\lambda}{}_{\mu}\, \underset{i}{\lambda}{}_{\nu;\rho}, \tag{50}
$$

and λ is a free dimensionless parameter. The energy is given by the surface integral (cf.[15])

$$
E \stackrel{\text{def.}}{=} lim_{\rho\to\infty} \int_{\rho=constant} \mathcal{U}_0{}^{0\alpha} n_\alpha dS, \tag{51}
$$

where n_α is the unit 3-vector normal to the surface element dS. The superpotential associated with the solution in the case of indefinite metric (33) is given by

$$\mathcal{U}_0{}^{0\alpha} = \frac{X^\alpha}{4\pi r^3}\left[2m - r^2 B'\right]. \tag{52}$$

Substituting (52) into (51), we obtain

$$E(r) = 2m - B' r^2. \tag{53}$$

It is clear from (53) that the energy associated with (33) is dependent on the arbitrary function $B(r)$ which does not appear in the line-element (31) but appears in the tetrad (33).

For the case of positive-definite solution (46), the superpotential is given by

$$\mathcal{U}_0{}^{0\alpha} = \frac{X^\alpha}{4\pi r^3}\left[k_2 + \frac{3c_1}{r}\right]. \tag{54}$$

Substituting (54) into (51), we obtain

$$E(r) = k_2 + \frac{3c_1}{r}. \tag{55}$$

5. Comparison and Discussion

In the present work we have studied the effect of signature change on the solutions of the field equations of NGR. As is pointed out in [12], NGR is a gravitational theory formulated using the AP-space. Any AP-structure is defined completely, in 4-dimensions, by a tetrad vector field subject to the AP-condition as stated in section 2. The importance of examining gravity theory written in this geometry is that tetrads play an important role in many aspects, e.g. they are used as fundamental variables in the attempts to quantize gravity [13]. So, one expects to get, using such theories, more information about gravity than those obtained from general relativity. This is one of the goals of the present study. In order to compare the results obtained with the corresponding results of general relativity, we have chosen the case having spherical symmetry for application. Also we have solved the field equations of NGR in free space, i.e. $T^{\mu\nu} = 0$.

As mentioned in the introduction, there are two philosophies behind the idea of signature change. The first assumes that the change of signature, from a positive -definite 4-dimensions to an indefinite (3+1)-dimensions, occurred at a certain epoch when the universe was younger, i.e. it occurred on a hypersurface of the space-time (cf. [5, 6, 7, 8]). The second philosophy assumes that the universe has a positive-definite metric, always and

everywhere, with 4-dimensions while our measurements and observations are carried out in (3+1)-dimensions with an indefinite metric [4]. Consequently, there is no need to impose the Lorentz signature on the metric (from which we formulate the differential equations) before solving these equations but this signature is to be imposed on the solutions of these equations, i.e. just before matching the results of the theory with measurements and observations. Although the first approach has some advantages, yet there is a singularity at the change surface [7]. Using the second philosophy, one can overcome this difficulty.

Calculations in the present work have been done following the second philosophy. To study the effect of signature change, we have solved the field equations of NGR in two different cases. In the first case we have imposed the Lorentz signature on the metric (imaginary tetrad) before formulating the differential equations as usually done in the scheme of GR. In the second case we have formulated and solved the differential field equations using a positive-definite metric (real tetrad), then we imposed the Lorentz signature on the solution obtained in order to compare the results with the well known physics of GR. The same procedure has been previously used in the case of general relativity [4]. The results obtained are similar to those obtained from general relativity. Table I summarizes these results and compares them with those previously obtained by Hayashi and Shirafuji [12].

Table 1: Comparison between the present solutions and Hayashi-Shirafuji Solution

	Case I	Case II	Hayashi-Shirafuji solution
Tetrad $\underset{0}{\lambda}_{\mu}$	imaginary, non-diagonal	real, non-diagonal	imaginary, diagonal
Metric	indefinite	+ve definite	indefinite
Number of solution	one	many	one
Schwarzschild solution	yes	yes	yes
Second-order skew tensors	some identically vanishing	all non-vanishing	all vanishing

From table I it is clear that, although the Schwarzschild metric is obtained in the three cases, these cases are associated with different sets of second-order skew tensors. As it is clear, the diagonal tetrad used by Hayashi and Shirafuji [12] does not produce any skew tensors of the second order. So, if some physics is to be attributed to these skew tensors, then the physical contents of the three cases, given in table II are quite different. The role of these second-order skew tensors is totally obscured in the case of Riemannian geometry, since such tensors are not defined in this geometry. For this reason general relativity was written in the AP-geometry when examining the effect of signature change [4]. It can be shown that the non-vanishing second-order skew tensors depend on the function $D(r)$, whose vanishing will give rise to the vanishing of those skew tensors. The role of such tensors can be clarified if field theory dealing with other interactions together with gravity. Such a situation was achieved in the case of the generalized field theory [10]. Referring to the discussion given in [4, 20, 21], it is shown that the non-vanishing skew tensors are related to the electromagnetic field.

On the other hand, in the solution given by (40) and (41) there are two different constants of integration c_1 and $\tilde{K}_2$. If we take $c_1 = 0$, the solution will give rise to the Schwarzschild

metric. Then, we can identify $\tilde{K}_2$ with the geometric mass of the source of the gravitational field. While, if $c_1 \neq 0$, then if we evaluate the metric (49) far from the source of the field, i.e. neglecting the term $O\frac{1}{(\tilde{R}^4)}$, the metric will be similar to Reissner-Nordström metric. This may indicate that the constant c_1 can be related to the electric charge of the source.

To investigate the structure of the two solutions (22), (27) and (40), (41) we make a physical application by calculating the energy associated with these two solutions using the superpotential potential method given by Mikhail et al. [15]. We transform the tetrad which gives the Schwarzschild solution to the Cartesian form to calculate the energy. As is clear from (53) the energy depends on the arbitrary function $B(r)$. If the asymptotic behavior is of $O(1/r)$ then the form of the energy becomes $E = 2m + \textit{some cont.}$, which in general is different from the general relativity in spite of the fact that the associated form of the line-element of these solution is the Schwarzschild. This is because the asymptotic form of this tetrad behaves like $O(1/\sqrt{r})$ [22]. We also calculate the energy associated with the second solution after transforming it to the Cartesian form using the superpotential method. As is clear from (55) we keep only up to $O(1/r)$. This may indicate that the structure of the two solutions different as we analyze above using the skew tensors argument.

Table 2: Comparison between the Energy of Different Solutions

Solution	Superpotential Method
Hayashi-Shirafuji diagonal solution	$E = m$
Present work, first solution Case I	$E = 2m+$ some cont. if $B(r) \sim O(1/r)$
Present work, second solution Case II	$E(r) = k_2 + \frac{3c_1}{r}$

It is clear from table II that the energy of Case II depends on r which is similar to Reissner-Nordström energy.

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